The Magnetization Inverse Problem on 2-D sources

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Abstract

Here we will present the research found on [3, 4]. This work is concerned with inverse potential problems with source term in divergence form. Such issues typically arise in source identification from field measurements for Maxwell's equations, in the quasi-static regime. They occur for instance in geomagnetism and paleomagnetism, as well as in several non-destructive testing problems; e.g., see [5, 8, 6] and their bibliographies. A model problem of our particular interest and which we will take as focus in this poster, is inverse scanning magnetic microscopy, as considered for instance in [2, 7, 1] to recover magnetization distributions of thin rock samples. However the considerations below are of a more general and abstract nature.

Research Groups

This work is part a collaborative project from the following research groups:

- E. A. Lima and B. Weiss of the MIT Department of Earth, Atmospheric, and Planetary Sciences.
- L. Baratchart, S. Chevillard, J. Leblond and D. Ponomarev of the FACTAS team in INRIA.
- D. Hardin of Vanderbilt University

We are interested in a mathematical framework for measurements obtained form a Scanning Magnetic Microscope (SMM) such as the instrument used by the research team in the MIT.

Overview

This problem consist of recovering the remanent magnetization M of a rock sample from measurements of its magnetic field B.

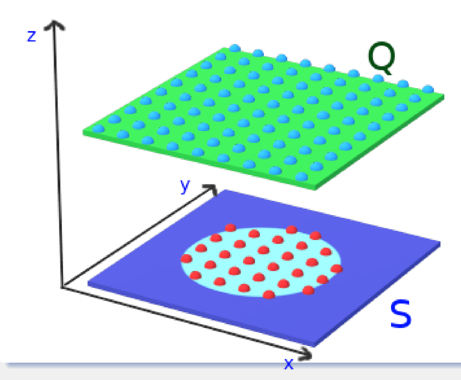
Using the magnetic potential Φ which satisfies

$$\begin{aligned} \mathrm{B} &= \mu_0 \left(\mathrm{M} - \operatorname{grad} \Phi\right) \\ \Delta \Phi &= \operatorname{div} \mathrm{M} \end{aligned} \tag{1}$$

we study the inverse problem for the recovery of M when this magnetization.

We use methods for recovering M based on total variation regularization.

SMM setup



Q will denote the set on which we take the measurement.

S will denote a super set of the magnetization support.

Magnetizations

The *magnetization* of an object is a density for the magnetic moment.

We will model magnetizations by vector valued Borel measures.

$\mathcal{M}(S)$ denotes the space of Borel measures supported on $S \subset \mathbb{R}^3$.

Each $\mathrm{M} \in \mathcal{M}(S)^3$ is of the form

$$d\mathrm{M} = \mathrm{u}_{\mathrm{M}} d|\mathrm{M}|,$$

where $|\mathrm{M}|$ is a positive Borel measure supported on S and $|\mathrm{u}_{\mathrm{M}}| = 1$ $|\mathrm{M}|$-a.e.

Then, the total variation norm of the measure M is defined by $\|\mathrm{M}\|_{TV} := |\mathrm{M}|(\mathbb{R}^3)$.

Forward problem

The field generated by a magnetization M at a point x not on the support of M is

$$\mathrm{B}(\mathrm{x}) = -\frac{\mu_0}{4\pi} \operatorname{grad} \Phi,$$

where Φ is defined for any point x not in the support of M as

$$\Phi(\mathrm{M})(\mathrm{x}) = \frac{1}{4\pi} \int \frac{\mathrm{x} - \mathrm{y}}{|\mathrm{x} - \mathrm{y}|^3} \cdot d\mathrm{M}(\mathrm{y}).$$

Consider a finite, positive Borel measure ρ with support contained in Q and let $\mathrm{A} : \mathcal{M}(S)^3 \to L^2(Q, \rho)$ be the operator defined by

$$\mathrm{A}(\mathrm{M})(\mathrm{x}) := \mathrm{v} \cdot \mathrm{B}(\mathrm{M})(\mathrm{x}), \qquad \mathrm{x} \in Q.$$

Difficulties with the inverse problem

A magnetization M is said to be **S-silent** if $\mathrm{B}(\mathrm{M})$ vanishes on $\mathbb{R}^3 \setminus S$.

Just by looking at the equation (1) we notice right away that all divergence free magnetizations are S-silent and thus the inverse problem is ill-posed.

However, this are not the only type of magnetizations that are silent: Let $S = \mathrm{B}$ be the closed unit Euclidean ball centred at the origin, S_0 its boundary, $\mathrm{M} = \mathcal{H}_2 \lfloor S_0$ (where $\mathcal{H}_2$ is area measure), $\mathrm{M} = (4\pi/3)^{-1} \mathrm{M} \mathrm{e}_1$ and $\mathrm{N} := \delta_0 \mathrm{e}_1$ where δ_0 is the Dirac delta at zero. Then, using the mean value theorem it follows that both have the same magnetic potential so, $\mathrm{M} - \mathrm{N}$ is S-silent, while $\mathrm{M} - \mathrm{N}$ is not divergence free. For that reason we need to restrict our attention to a particular type of sets:

We will call a closed set $S \subset \mathbb{R}^3$ a **slender set** if it is 2-D and each connected component C of $\mathbb{R}^3 \setminus S$ satisfies $\mathcal{L}_3(C) = \infty$.

In particular, planes are slender sets

Results for the restriction of the problem

Lemma

Let $S \in \mathbb{R}^3$ be closed and slender. Then M is S-silent if and only if $div\,\mathrm{M} = 0$.

The work in [3] let us decompose divergence free magnetization into a continuous sum of path integration of Jordan curves, which in turn give us the following result, which applies to a family of real samples that the team in the MIT actually work on:

Theorem

Suppose $S \subset \mathbb{R}^3$ is a closed, slender set. If $\mathrm{M} \in \mathcal{M}(S)^3$ satisfies either:

- *having a* sparse *support,*
- *u_{M} is constant a.e. with respect to $|\mathrm{M}|$*

Then, for any $\mathrm{N} \in \mathcal{M}(S)^3$ with $\mathrm{N} - \mathrm{M}$ being S-silent, we have that $\|\mathrm{N}\|_{TV} > \|\mathrm{M}\|_{TV}$ unless $\mathrm{N} = \mathrm{M}$.

This result is telling us that, taking a regularization scheme using the TV-norm should give us back the original magnetization provided the regularization parameter is small enough.

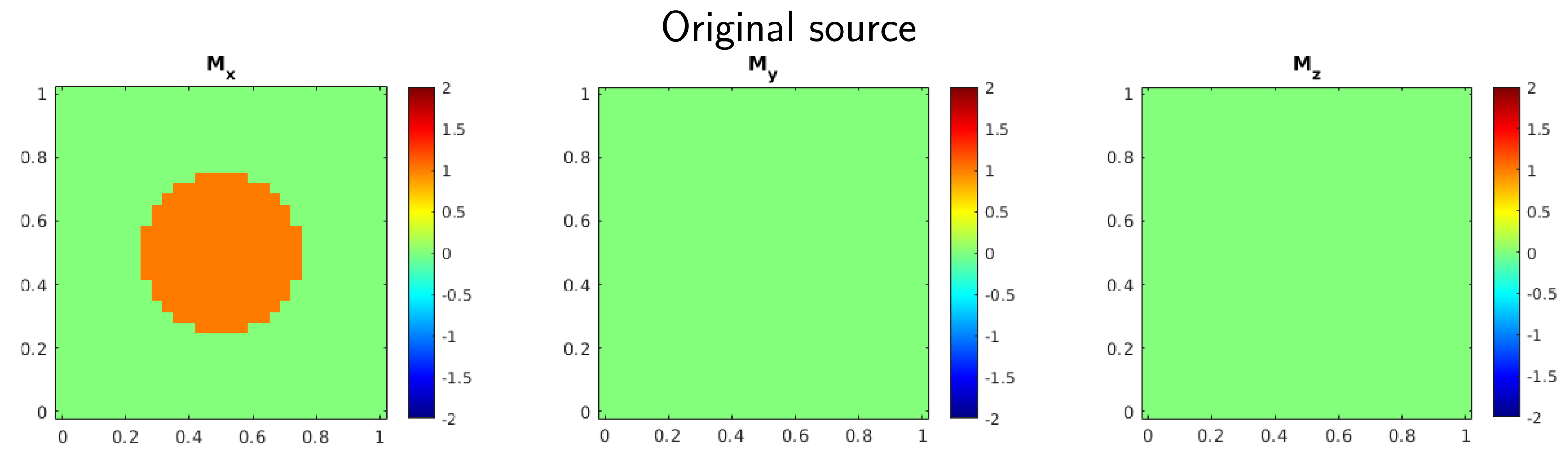
The group-LASSO regularization technique

Using the group LASSO (least absolute shrinkage and selection operator)

$$\arg\min_{\mathrm{m}} \|b - A\mathrm{m}\|_2^2 + \lambda \sum_i \|\mathrm{m}_i\|_2.$$

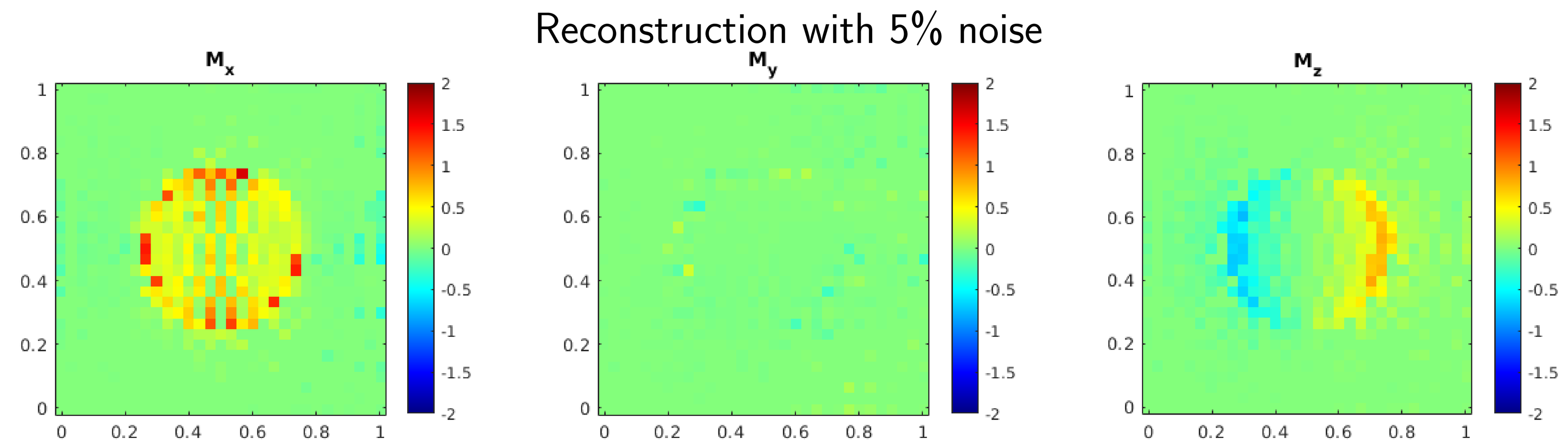
on a disk that is magnetized in the x-direction:

Original source



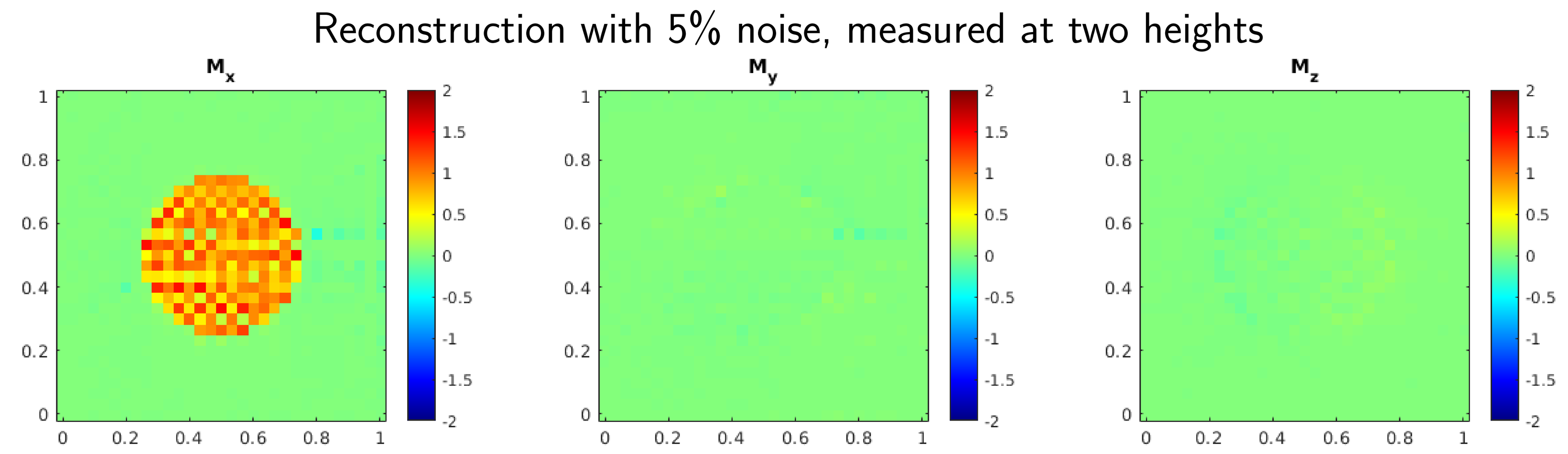
and adding 5% noise to the measurements, we obtain the following

Reconstruction with 5% noise



Although the support is recovered we still get some extra components on the z-direction. Fortunately, this issue is lessen if we assume that we are taking measurements on two planes:

Reconstruction with 5% noise, measured at two heights



Conclusion and further work

Even though the original set up of the experiment is not ideal for the inversion of the problem, we showed that if it was possible to measure in two planes the results will improve substantially. Unfortunately, this presents a logistical challenge for the MIT due to the type of microscope.

This work has motivated D. Ponomarev, J. Leblond and myself to start looking at the possibility of extending the given samples to a second plane to improve the reconstructions.

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