UNIQUENESS RESULT FOR TV-NORM REGULARIZED INVERSE PROBLEMS WITH SOURCE TERM IN DIVERGENCE FORM

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ABSTRACT. Inverse source problems in divergence form consist in finding a vector field with prescribed support S, whose divergence is the Laplacian of some observed potential. In this paper, we assume the unknown vector field is a vector-valued measure, and we study the corresponding least square inversion problems, regularized by penalizing the total variation, without discretizing the criterion nor the unknown. We prove that this problem that this problem has a unique minimizer in the case where S is a so-called slender set; i.e., it has zero Lebesgue measure and each connected component of its complement has infinite Lebesgue measure. The proof dwells on Smirnov's decomposition of divergence-free measures [28] that quickly turns the paper into a measure-geometric one.

1. INTRODUCTION

Inverse source problems in divergence form consist, roughly speaking, in finding a vector field with prescribed support whose divergence is the Laplacian of some observed potential. They arise in various contexts, for instance Geomagnetism and Paleomagnetism, or else Medical Imaging from Electro-Encephalography (EEG) [8, 22, 3, 17, 30, 27, 23]. Such problems are severely ill-posed and solutions highly non-unique in general, making regularization techniques an essential aspect of every approach. In this paper, we assume the unknown vector field is a vector-valued measure and we study the corresponding least square inversion problems, regularized by penalizing the total variation, without discretizing the criterion nor the unknown.

This is in contrast with approaches where the model gets discretized in the first place, so as to deal only with a finite-dimensional optimization scheme [2, 18, 9, 24]. The unsatisfactory side of doing so is that, when the dimension is infinite, discretization involves an approximation of the underlying operators that interacts in a convoluted manner with regularization (which is another kind of approximation) and may affect inversion in adverse ways. This is documented in several works at a general functional level and from a statistical viewpoint [19, 11, 1], moreover thorough discussions are available for certain elliptic equations with Tikhonov-like regularization [25, 29, 15], but the phenomenon seems much less studied when the forward operator is non-injective. In fact, when the forward operator has nontrivial kernel, discretization typically turns it into some injective but ill-conditioned matrix, whereas the very structure of the kernel could be used to design regularization schemes and define suitable notions of sparsity or of minimal solutions. The situation that we consider is exactly of this type.

Our setting is a particular instance of the one in [10], but we will prove more than could be deduced from that general reference when the forward operator is not injective and the so-called source condition does not hold, as is the case here. Altogether, we shall discuss recovery of some unknown $\mathbb{R}^N$-valued measure carried by a compact subset S of $\mathbb{R}^N$ (the sample), from knowledge of (a directional derivative of) the Newton potential of its divergence on another compact subset Q of $\mathbb{R}^N$ (the measurement slab). In applications that we have in mind, one would set $N = 3$ but this does not simplify the matter. Typical examples stem from inverse magnetization problems arising in Paleomagnetism, and this will be a model case for us [20, 16], but the paper makes room for more general frameworks. That is to say, for fixed $\lambda > 0$ and $f \in L^2(Q)$ with $Q \subset \mathbb{R}^N$, we study minimizers of the functional

$$\boldsymbol{\mu} \mapsto \|f - \mathfrak{A}\boldsymbol{\mu}\|^2_{L^2(Q)} + \lambda\|\boldsymbol{\mu}\|_{TV} \tag{1}$$

when $\boldsymbol{\mu}$ ranges over $\mathbb{R}^N$-valued measures supported on a given compact set $S \subset \mathbb{R}^N$, disjoint from Q, where $\mathfrak{A}$ is the operator mapping $\boldsymbol{\mu}$ to $v \cdot \nabla u$ with v a fixed unit vector and the potential u solves $\delta u = \mathrm{div}\boldsymbol{\mu}$, while $\|\boldsymbol{\mu}\|_{TV}$ is the total variation of the measure $\boldsymbol{\mu}$.

Total variation regularization for such problems was considered in [6, 5], and shown there to be consistent for recovering certain classes of measures that may be called sparse; e.g., those with finite (more generally: purely 1-unrectifiable) support. These results assume that the sample S is a so-called *slender* set; *i.e.*, it has zero Lebesgue measure and each connected component of its complement has infinite Lebesgue measure. As is customary in the literature on identification and inverse problems, *consistency* means that any sequence of minimizers of (1) converges to the unknown measure $\boldsymbol{\mu}_0$ such that $\mathfrak{A}\boldsymbol{\mu}_0 = f_0$ when both the regularization parameter λ and the measurement error $\|f - f_0\|_{L^2(Q)}$ go to zero, assuming that the latter decays faster than a suitable function (in our case the square root) of the former (the Morozov discrepancy rule). Here, the convergence is stronger than the usual weak-$*$ convergence of measures, for in addition to the latter the total variation along the sequence converges to the total variation of the limit; *i.e.*, despite the amount of cancellation that may occur in the weak-$*$ limit, no mass is "lost" in the process.

In the particular case where S is contained in a plane, it was further proven in [5] that (1) has a unique minimizer, whatever the data f. In the present work, we extend this result to the case where S is slender, but not necessarily contained in a plane. That is to say: in the slender case, the regularized criterion has a unique minimizer. On the one hand, the minimization problem under consideration is convex but not strictly convex and the property just mentioned is not obvious: its proof dwells on Smirnov's decomposition of divergence-free measures [28] that quickly turns the paper into a measure-geometric one. On the other hand, we want to stress its relevance from an inverse problem perspective as follows. In applications, the regularized criterion (1) must be discretized into a finite-dimensional one, and uniqueness of a minimizer will ensure convergence of the minimum places of the discretized criteria to the minimizer of (1), for fixed value of the regularization parameter λ, when the discretization gets refined over and over. It is so because reasonable discretization schemes will produce subsequential convergence of discrete minimizers to a minimizer of (1) (from any subsequence one can extract a convergent subsequence), and then uniqueness of the limit entails convergence of the whole sequence. In this paper we do not discuss the derivation of such convergent discretization schemes; the authors will report on such issues in [4]. Here, we merely stress that the uniqueness result to be proven below allows one to design consistent recovery schemes for sparse inverse source problems in divergence form, *via* a sequence of discrete, finite-dimensional optimization problems. Note that the existence of a sparse solution in the sense of [6, 5] is here essential to ascertain consistency of a recovery scheme based on minimizing (1), but that such an infinite-dimensional notion of sparsity is completely different from the sparse character of minimizers of the discretized criterion (that can essentially always be achieved, since discrete measures are weak$*$ dense in the space of measures, but implies nothing on the sparsity of the measure to be recovered).

2. Preliminaries and notation

Let N be a positive integer and S a compact subset of $\mathbb{R}^N$. We let $C(S)$ denote the real-valued continuous functions on S and $\mathcal{M}(S)$ the finite signed Borel measures on $\mathbb{R}^N$ supported on S. Equipped with the *total variation norm* $\|\cdot\|_{TV}$, $\mathcal{M}(S)^N$ is a Banach space, dual to $C(S)^N$ under the pairing $\langle \boldsymbol{\mu}, \boldsymbol{\varphi}\rangle = \sum_{i=1}^N \int \varphi_i d\mu_i$, with μ_i and φ_i to mean the components of $\boldsymbol{\mu}$ and $\boldsymbol{\varphi}$ respectively. We also consider $\mathcal{M}(S)^N$ as a Fréchet space endowed with the weak-star topology, the dual of which is $C(S)^N$ [26, Theorem 3.10].

Furthermore, for a $\boldsymbol{\mu} \in \mathcal{M}(S)^N$, we will let $|\boldsymbol{\mu}|$ denote its *total variation measure* and $\boldsymbol{u}_{\boldsymbol{\mu}}$ the Radon-Nikodym derivative of $\boldsymbol{\mu}$ with respect to $|\boldsymbol{\mu}|$, so we have, $d\boldsymbol{\mu} = \boldsymbol{u}_{\boldsymbol{\mu}} d|\boldsymbol{\mu}|$.

As a short hand, for a scalar measure $\mu \in \mathcal{M}(S)$ and a vector function $\boldsymbol{\varphi} \in C(S)^N$, we will let $\langle \mu, \boldsymbol{\varphi}\rangle := \sum_{i=1}^N \langle \mu, \varphi_i\rangle \mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th canonical unit vector.

For $a < b$, we let $\mathrm{Lip}([a,b])$ denote the space of Lipschitz maps $\gamma : [a,b] \to \mathbb{R}$. We call the elements of $\mathrm{Lip}([a,b])^N$ *parametrized rectifiable curves*. We further let $\mathrm{Lip}_1([a,b])$ denote the subset of Lipschitz maps with Lipschitz constant less than or equal to 1. From the Arzela-Ascoli theorem [12, Theorem 4.7], it follows that $\mathrm{Lip}_1([a,b])$ is locally compact when equiped with the uniform norm.

For the following definitions, fix a $\boldsymbol{\gamma} \in \mathrm{Lip}([a,b])^N$; if we put $\#(\boldsymbol{\gamma}, x)$ for the cardinality (finite or infinite) of the preimage $\boldsymbol{\gamma}^{-1}(x)$, then the length $\ell(\boldsymbol{\gamma})$ of $\boldsymbol{\gamma}$ is

$$\ell(\boldsymbol{\gamma}) := \int_a^b |\boldsymbol{\gamma}'(t)|\, dt = \int \#(\boldsymbol{\gamma}, x)\, d\eta^1(x), \tag{2}$$

where $\boldsymbol{\gamma}'$ denotes the derivative of $\boldsymbol{\gamma}$ and η^1 means 1-dimensional Hausdorff measure normalized to coincide with standard arc length, and the second equality follows from the area formula [14, 3.2.3]. We also define $\boldsymbol{\pi}(\boldsymbol{\gamma}) = \boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{M}(\mathbb{R}^n)^n$ by

$$\langle \boldsymbol{\pi}_{\boldsymbol{\gamma}}, \boldsymbol{g}\rangle := \int_a^b \boldsymbol{g}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) dt = \int_\Gamma \left(\sum_{t \in \boldsymbol{\gamma}^{-1}(x)} \boldsymbol{g}(x) \cdot \boldsymbol{\gamma}'(t) \right) d\eta^1(x), \qquad \boldsymbol{g} \in C(\mathbb{R}^n)^n, \tag{3}$$

where the second equality again follows from the area formula.

For each $\ell > 0$, consider the collection $\mathcal{C}_\ell$ of those $\boldsymbol{\pi}_{\boldsymbol{\gamma}}$ associated to a parametrized rectifiable curve $\boldsymbol{\gamma}$ of length ℓ that satisfy $\|\boldsymbol{\pi}_{\boldsymbol{\gamma}}\|_{TV} = \ell$, and equip $\mathcal{C}_\ell$ with the weak-star topology. Since $\boldsymbol{\pi}_{\boldsymbol{\gamma}}$ is absolutely continuous with respect to the restriction $\eta^1 \lfloor \boldsymbol{\gamma}([a,b])$, we may consider the Radon-Nikodym derivative $\boldsymbol{\tau}$ of $\boldsymbol{\pi}_{\boldsymbol{\gamma}}$ with respect to η^1 and we remark in view of (3) that $\boldsymbol{\tau}(x) = \sum_{t \in \boldsymbol{\gamma}^{-1}(x)} \boldsymbol{\gamma}'(t)$ for η^1-a.e. $x \in \boldsymbol{\gamma}[a,b]$. Therefore, if $\boldsymbol{\gamma} \in \mathrm{Lip}_1([a,b])$, then $\boldsymbol{\gamma} \in \mathcal{C}_{\ell(\boldsymbol{\gamma})}$ if and only if $|\boldsymbol{\tau}(x)| = \#(\boldsymbol{\gamma}, x)$ for a.e. $x \in \boldsymbol{\gamma}([a,b])$, by (2); that is, if and only if $\boldsymbol{u}_{\boldsymbol{\pi}_{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}(t)) = \boldsymbol{\gamma}'(t)$ for a.e. $t \in [0,\ell]$ (entailing that $\boldsymbol{\gamma}'(t)$ depends only on $\boldsymbol{\gamma}(t)$ for almost every t). One can check that for every $\boldsymbol{\rho} \in \mathcal{C}_\ell$, there is a unit length parametrized rectifiable curve $\boldsymbol{\gamma}$ such that $\boldsymbol{\rho} = \boldsymbol{\pi}_{\boldsymbol{\gamma}}$.

Now, suppose that $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^N)^N$ is a solenoid, *i.e.* that $\nabla \cdot \boldsymbol{\mu} = 0$; *i.e.*, $\boldsymbol{\mu}$ is divergence-free as a distribution. Then, it follows from [28, Theorem A] that $\boldsymbol{\mu}$ can be decomposed into elements from $\mathcal{C}_\ell$, for any $\ell > 0$. That is, there is a positive finite Borel measure Ξ on $\mathcal{C}_\ell$ such that Ξ-a.e. $\boldsymbol{\pi}_{\boldsymbol{\gamma}}$ is supported in $\operatorname{supp} \boldsymbol{\mu}$, and for each Borel set $B \subset \mathbb{R}^N$:

$$\boldsymbol{\mu}(B) = \int_{\mathcal{C}_\ell} \boldsymbol{\pi}_{\boldsymbol{\gamma}}(B)\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}), \qquad |\boldsymbol{\mu}|(B) = \int_{\mathcal{C}_\ell} |\boldsymbol{\pi}_{\boldsymbol{\gamma}}|(B)\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}). \tag{4}$$ [1]

3. Results on divergence free measures

We shall need the continuity of the map $\boldsymbol{\pi}$ just defined:

Lemma 3.1. *The map $\boldsymbol{\pi} : \mathrm{Lip}_1([a,b])^N \to \mathcal{M}(\mathbb{R}^n)^n$ is continuous when $\mathcal{M}(\mathbb{R}^n)^n$ is endowed with the weak-star topology.*

Proof. Let $\{\boldsymbol{\gamma}_n\}_{n \in \mathbb{N}^*} \subset \mathrm{Lip}_1([a,b])^N$ converge uniformly to $\boldsymbol{\gamma}_0 \in \mathrm{Lip}_1([a,b])^N$. Define the map $\pi : \mathrm{Lip}_1([a,b])^N \to \mathcal{M}([a,b])^N$ for any $\boldsymbol{\gamma} \in \mathrm{Lip}_1([a,b])^N$ by $d\pi(\boldsymbol{\gamma})(t) := \boldsymbol{\gamma}'(t)dt$, where dt is the differential of 1-dimensional Lebesgue measure. As Lipschitz functions are absolutely continuous, for any subinterval $(s,t) \subset [a,b]$ we get $\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(s) = \pi(\boldsymbol{\gamma})((s,t))$ and similarly for closed intervals. Thus, $\pi(\boldsymbol{\gamma}_n)(O) \to \pi(\boldsymbol{\gamma})(O)$ for each relative open subset O of $[a,b]$, therefore $\pi(\boldsymbol{\gamma}_n) \to \pi(\boldsymbol{\gamma}_0)$ weak-star as $n \to \infty$ because the $\pi(\boldsymbol{\gamma}_n)$ have uniformly bounded total variation, hence $\{\pi(\boldsymbol{\gamma}_n)\}_{n \in \mathbb{N}^*}$ is relatively weak-star compact in $\mathcal{M}(\mathbb{R}^n)^n$ by the Banach-Alaoglu theorem, but $\pi(\boldsymbol{\gamma}_0)$ is the only possible limit point.

Now, pick $\boldsymbol{f} \in C(S)$ and $\varepsilon > 0$. Let $M, \delta > 0$ and the integer m be such that, for any $n > m$, one has:

(i) $|\langle \pi(\boldsymbol{\gamma}_n) - \pi(\boldsymbol{\gamma}_0),\ \boldsymbol{f} \circ \boldsymbol{\gamma}_0 \rangle| < \frac{\varepsilon}{2}$,
(ii) $\|\boldsymbol{\gamma}_0 - \boldsymbol{\gamma}_n\|_\infty < \delta$,
(iii) for any $x, y \in S$ with $|x - y| < \delta$, it holds that $|\boldsymbol{f}(x) - \boldsymbol{f}(y)| < \frac{\varepsilon}{2(b-a)}$.

Thus, remembering that $|\boldsymbol{\gamma}'| \leq 1$ for any $\boldsymbol{\gamma} \in \mathrm{Lip}_1([a,b])^N$, we have that

$$|\langle \boldsymbol{\pi}_{\boldsymbol{\gamma}_n} - \boldsymbol{\pi}_{\boldsymbol{\gamma}_0},\ \boldsymbol{f} \rangle| = \left| \int_a^b \boldsymbol{f}(\boldsymbol{\gamma}_n(t)) \cdot \boldsymbol{\gamma}_n'(t)\, dt - \int_a^b \boldsymbol{f}(\boldsymbol{\gamma}_0(t)) \cdot \boldsymbol{\gamma}_0'(t)\, dt \right|$$

[1] In [28], Smirnov states his decomposition in terms of integration against test functions, but ours easily follows from his by approximation of characteristic functions.

$$\leq \left| \int_a^b \boldsymbol{f}(\boldsymbol{\gamma}_n(t)) \cdot \boldsymbol{\gamma}_n'(t)\ dt - \int_a^b \boldsymbol{f}(\boldsymbol{\gamma}_0(t)) \cdot \boldsymbol{\gamma}_n'(t)\ dt \right|$$
$$+ \left| \int_a^b \boldsymbol{f}(\boldsymbol{\gamma}_0(t)) \cdot \boldsymbol{\gamma}_n'(t)\ dt - \int_a^b \boldsymbol{f}(\boldsymbol{\gamma}_0(t)) \cdot \boldsymbol{\gamma}_0'(t)\ dt \right|$$
$$\leq \int_a^b |\boldsymbol{f}(\boldsymbol{\gamma}_n(t)) - \boldsymbol{f}(\boldsymbol{\gamma}_0(t))|\ dt + |\ \langle\ \pi(\boldsymbol{\gamma}_n) - \pi(\boldsymbol{\gamma}_0),\ \boldsymbol{f} \circ \boldsymbol{\gamma}_0\ \rangle\ | < \varepsilon.$$

Therefore, the map $\boldsymbol{\pi}$ is uniform to weak-star continuous. □

Remark 1. Taking $\boldsymbol{\gamma} \in \mathrm{Lip}([a,b])^N$ with $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{C}_\ell$, note that $\nabla\cdot\boldsymbol{\pi}_{\boldsymbol{\gamma}} = \delta_{\boldsymbol{\gamma}(b)} - \delta_{\boldsymbol{\gamma}(a)}$, i.e. the difference of the Dirac's delta at $\boldsymbol{\gamma}(b)$ minus the one at $\boldsymbol{\gamma}(a)$. Also, if we take $\tilde{\boldsymbol{\gamma}} \in \mathrm{Lip}([\tilde{a},\tilde{b}])^N$ such that $\boldsymbol{\pi}_{\boldsymbol{\gamma}} = \boldsymbol{\pi}_{\tilde{\boldsymbol{\gamma}}}$, we get that, if $\boldsymbol{\gamma}(a) \neq \boldsymbol{\gamma}(b)$, then, $\boldsymbol{\gamma}(a) = \tilde{\boldsymbol{\gamma}}(\tilde{a})$ and $\boldsymbol{\gamma}(b) = \tilde{\boldsymbol{\gamma}}(\tilde{b})$. In particular, to any $\boldsymbol{\nu} \in \mathcal{C}_\ell$ which is not divergence-free, there is a unique Lipschitz $\boldsymbol{\gamma} : [0,\ell] \to \mathbb{R}^N$ with $|\boldsymbol{\gamma}'|(t) = 1$ for a.e. t such that $\boldsymbol{\eta} = \boldsymbol{\pi}_{\boldsymbol{\gamma}}$. Finally, given any $t \in [a,b]$ and recalling that the uniform norm makes $\mathrm{Lip}_1([a,b])^N$ a compact space, we get since the evaluation map $\boldsymbol{\gamma} \to \boldsymbol{\gamma}(t)$ is continuous that it is also closed (images of closed sets are closed) and, considering small translations of a curve in arbitrary directions, we see that it is also an open map.

The following lemma is fundamental to our purposes. The main idea of the proof is to find the set of curves in the decomposition (4) that we can paste together without exiting the support of $\boldsymbol{\mu}$, and to show that the image of curves not belonging to this set have zero $|\boldsymbol{\mu}|$-measure

Proposition 3.2. *Let $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^N)^N$ be a divergence free measure and let X be a set of full $|\boldsymbol{\mu}|$-measure.*

For $|\boldsymbol{\mu}|$-a.e. $x \in X$, there exists a Lipschitz continuous function $\boldsymbol{f}_x : \mathbb{R} \to \mathbb{R}^N$ such that $\boldsymbol{f}_x(t_0) = x$ for some $t_0 \in \mathbb{R}$, and for a.e. $t \in \mathbb{R}$, $\boldsymbol{f}_x(t) \in X$ and $\boldsymbol{f}_x'(t) = \boldsymbol{u}_{\boldsymbol{\mu}}(\boldsymbol{f}_x(t))$.

Proof. Without loss of generality, we assume when $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{C}_1$ that $\boldsymbol{\gamma} \in \mathrm{Lip}([0,1])^N$ and that it is parametrized by arc length. There exists a positive Borel measure Ξ on $\mathcal{C}_1$ such that (4) is satisfied. Using (4) twice and Fubini's theorem, we obtain

$$\int_{\mathcal{C}_1} \int \boldsymbol{u}_{\boldsymbol{\mu}} \cdot d\boldsymbol{\pi}_{\boldsymbol{\gamma}}\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}) = \int \boldsymbol{u}_{\boldsymbol{\mu}} \cdot d\boldsymbol{\mu} = \|\boldsymbol{\mu}\|_{TV} = |\boldsymbol{\mu}|(\mathbb{R}^3) = \int_{\mathcal{C}_1} \|\boldsymbol{\pi}_{\boldsymbol{\gamma}}\|_{TV}\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}).$$

Thus, using the Cauchy–Schwartz inequality, it follows that for Ξ-a.e. $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{C}_1$ the equality $\boldsymbol{u}_{\boldsymbol{\mu}} = \boldsymbol{u}_{\boldsymbol{\pi}_{\boldsymbol{\gamma}}}$ holds $|\boldsymbol{\pi}_{\boldsymbol{\gamma}}|$-a.e. Hence, by definition of $\boldsymbol{\pi}_{\boldsymbol{\gamma}}$, we get for Ξ-a.e. $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{C}_1$ that the identity $\boldsymbol{u}_{\boldsymbol{\mu}}(\boldsymbol{\gamma}(t)) = \boldsymbol{\gamma}'(t)$ holds for a.e. $t \in [0,1]$. Now, using the right hand equality of (4) to calculate $|\boldsymbol{\mu}|(\mathbb{R}^3 \setminus X)$ we also get that for Ξ-a.e. $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{C}_1$, $|\boldsymbol{\pi}_{\boldsymbol{\gamma}}|(\mathbb{R}^3 \setminus X) = 0$, and hence, for a.e. $t \in [0,1]$, $\boldsymbol{\gamma}(t) \subset X$. Let then

$$\tilde{\mathcal{X}} := \{\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{C}_1 :\ \text{for a.e. } t \in [0,1],\ \boldsymbol{u}_{\boldsymbol{\mu}}(\boldsymbol{\gamma}(t)) = \boldsymbol{\gamma}'(t) \text{ and } \boldsymbol{\gamma}(t) \subset X\},$$

which has full Ξ-measure by what precedes. By the regularity of Borel measures on metric spaces (see for example [7, Theorem 1.1]), we can find $\mathcal{X} \subset \tilde{\mathcal{X}}$ which is a countable union of nested closed sets having full measure as well.

Put $\mathcal{C}^{\mathcal{J}} := \{\boldsymbol{\pi_\gamma} \in \mathcal{C}_1 : \boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1)\}$ for the set of closed curves in $\mathcal{C}_1$. We define the beginning and ending points for a measure $\boldsymbol{\pi_\gamma}$ as follows. Denoting the evaluation functions at 0 and 1 by $\boldsymbol{b}$ and $\boldsymbol{e}$ respectively (for beginning and ending points), consider the set functions $\mathfrak{b}$ and $\mathfrak{e}$ from subsets of $\mathcal{C}_1$ to subsets of $\mathbb{R}^N$ given by $\mathfrak{b} := \boldsymbol{b} \circ \boldsymbol{\pi}^{-1}$ and $\mathfrak{e} := \boldsymbol{e} \circ \boldsymbol{\pi}^{-1}$. By remark 1 it follows that

$$\mathfrak{b}(\{\boldsymbol{\pi_\gamma}\}) = \begin{cases} \{\boldsymbol{\gamma}(0)\} & \text{for } \boldsymbol{\pi_\gamma} \in \mathcal{C}_1 \setminus \mathcal{C}^{\mathcal{J}} \\ \boldsymbol{\gamma}([0,1]) & \text{otherwise} \end{cases} \quad \text{and} \quad \mathfrak{e}(\{\boldsymbol{\pi_\gamma}\}) = \begin{cases} \{\boldsymbol{\gamma}(1)\} & \text{for } \boldsymbol{\pi_\gamma} \in \mathcal{C}_1 \setminus \mathcal{C}^{\mathcal{J}} \\ \boldsymbol{\gamma}([0,1]) & \text{otherwise.} \end{cases}$$

Abusing notation slightly, we shall write $\mathfrak{b}(\boldsymbol{\pi_\gamma}) := \boldsymbol{\gamma}(0)$ and $\mathfrak{e}(\boldsymbol{\pi_\gamma}) := \boldsymbol{\gamma}(1)$ for $\boldsymbol{\pi_\gamma} \in \mathcal{C}_1 \setminus \mathcal{C}^{\mathcal{J}}$. Also, by remark 1, we get for $\boldsymbol{\pi_\gamma} \in \mathcal{C}_1 \setminus \mathcal{C}^{\mathcal{J}}$ that $\nabla\cdot \boldsymbol{\pi_\gamma} = \delta_{\mathfrak{e}(\boldsymbol{\pi_\gamma})} - \delta_{\mathfrak{b}(\boldsymbol{\pi_\gamma})}$, hence $\nabla\cdot$ is linear and continuous on the span of $\mathcal{C}_1$ with values in $\mathcal{M}(S)^N$ (equiped wih the weak-star topology) and thus, both $\mathfrak{b}|_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}}$ and $\mathfrak{e}|_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}}$ are weak-star continuous. Then, we can define $\beta, \epsilon \in \mathcal{M}(\mathbb{R}^N)^+$ (positive measures on $\mathbb{R}^N$), such that for any $\phi \in C(\mathbb{R}^N)$,

$$\langle \beta, \phi\rangle := \int_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}} \langle \delta_{\mathfrak{b}(\boldsymbol{\pi_\gamma})}, \phi\rangle \, d\Xi(\boldsymbol{\pi_\gamma}) \quad \text{and} \quad \langle \epsilon, \phi\rangle := \int_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}} \langle \delta_{\mathfrak{e}(\boldsymbol{\pi_\gamma})}, \phi\rangle \, d\Xi(\boldsymbol{\pi_\gamma})$$

By definition of the weak divergence $\nabla\cdot\boldsymbol{\mu} = \epsilon - \beta$, and since $\boldsymbol{\mu}$ is divergence free we get that $\epsilon = \beta$. For a closed set $B \subset \mathbb{R}^N$, we get by outer regularity of finite Borel measures on $\mathbb{R}^N$ and Urysohn's lemma that

$$\beta(B) = \int_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}} \delta_{\mathfrak{b}(\boldsymbol{\pi_\gamma})}(B) \, d\Xi(\boldsymbol{\pi_\gamma}) \quad \text{and} \quad \epsilon(B) = \int_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}} \delta_{\mathfrak{e}(\boldsymbol{\pi_\gamma})}(B) \, d\Xi(\boldsymbol{\pi_\gamma}).$$

Moreover, by inner regularity of these measures, the equality above holds for any Borel set B. Thus, for B a Borel set,

$$\beta(B) = \int_{\mathcal{C}_1\setminus\mathcal{C}^{\mathcal{J}}} \chi_{\mathfrak{b}^{-1}(B)}(\boldsymbol{\pi_\gamma}) \, d\Xi(\boldsymbol{\pi_\gamma}) = \Xi(\mathfrak{b}^{-1}(B) \setminus \mathcal{C}^{\mathcal{J}}) \text{ and } \epsilon(B) = \Xi(\mathfrak{e}^{-1}(B) \setminus \mathcal{C}^{\mathcal{J}}). \tag{5}$$

We next define sets $\mathcal{B}^F, \mathcal{E}^F \subset \mathcal{X}$; roughly speaking, $\mathcal{X} \setminus \mathcal{B}^F$ represents the set of all curves in $\mathcal{X}$ that can be indefinitely extended backwards using other curves from $\mathcal{X}$ and, analogously, $\mathcal{X} \setminus \mathcal{E}^F$ corresponds to curves in $\mathcal{X}$ that can be indefinitely extended forwards. This extension may be done using infinitely many times the same curve, if necessary. First note that by lemma 3.1 and remark 1, for any set $\mathcal{U} \in \mathcal{C}_1$ that is either open or closed, $\mathfrak{e}(\mathcal{U})$ is also open or closed respectively. Now, start by defining $\mathcal{B}^0 := \{\boldsymbol{\pi_\gamma} \in \mathcal{C}_1 : \mathfrak{b}(\boldsymbol{\pi_\gamma}) \not\subset \mathfrak{e}(\mathcal{X})\} = \mathfrak{b}^{-1}(\mathbb{R}^N \setminus \mathfrak{e}(\mathcal{X}))$ (corresponding to curves that do not begin where a curve of $\mathcal{X}$ ends) and note that this set is a countable intersection of nested open sets, and thus, a Borel set; proceeding inductively for each integer $n > 0$, assume that $\mathcal{B}^{n-1}$ is a countable intersection of nested open sets and let

$$\mathcal{B}^n := \mathfrak{b}^{-1}(\mathfrak{e}(\mathcal{B}^{n-1}) \setminus \mathfrak{e}(\mathcal{X} \setminus \mathcal{B}^{n-1}))$$

(corresponding to curves that begin at a point that is an endpoint of a curve of $\mathcal{B}^{n-1}$ and of no curve in $\mathcal{X}$ that does not belong to $\mathcal{B}^{n-1}$). Since $\mathcal{B}^{n-1}$ is a countable intersection of nested open sets, so are $\mathfrak{e}(\mathcal{B}^{n-1})$ and $\mathbb{R}^N \setminus \mathfrak{e}(\mathcal{X} \setminus \mathcal{B}^{n-1})$. Therefore, $\mathcal{B}^n$ defined above is a countable intersection of nested open sets as well. Finally, let $\mathcal{B}^F = \bigcup_n \mathcal{B}^n$ and define analogously, exchanging $\mathfrak{b}$ and $\mathfrak{e}$, the set $\mathcal{E}^F$. Note that both $\mathcal{B}^F$ and $\mathcal{E}^F$ are Borel subsets of $\mathcal{C}_1$.

Since $\boldsymbol{u}_{\boldsymbol{\pi}_{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}(t)) = \boldsymbol{\gamma}'(t)$ for a.e. $t \in [0,1]$, for each $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{X} \setminus (\mathcal{B}^F \cup \mathcal{E}^F)$ we can extend $\boldsymbol{\gamma}$ to a Lipschitz continuous function $\boldsymbol{f} : \mathbb{R} \to \operatorname{supp}(\boldsymbol{\mu})$ such that $\boldsymbol{f}'(t) = \boldsymbol{u}_{\boldsymbol{\mu}}(\boldsymbol{f}(t))$ a.e. on $\mathbb{R}$. Now, let $\mathcal{Y} \subset \mathcal{X} \setminus (\mathcal{B}^F \cup \mathcal{E}^F)$ be a countable union of closed sets such that $\Xi(\mathcal{Y}) = \Xi(\mathcal{X} \setminus (\mathcal{B}^F \cup \mathcal{E}^F))$; such a $\mathcal{Y}$ exists by inner regularity of Borel measures on metric spaces. Equip the product space $\mathrm{Lip}_1([0,1])^N \times [0,1]$ with the norm $\|\cdot\|_\infty + |\cdot|$. Then, the function $\mathfrak{Ev} : \mathrm{Lip}_1([0,1])^N \times [0,1]$ defined by $\mathfrak{Ev}(\boldsymbol{\gamma}, t) := \boldsymbol{\gamma}(t)$, is continuous. Next, $\boldsymbol{\pi}^{-1}(\mathcal{Y})$ is a countable union of closed sets which, since $\mathrm{Lip}_1([0,1])^N$ is locally compact, can be written as a countable union of compact sets. Thus, the set $I := \mathfrak{Ev}(\boldsymbol{\pi}^{-1}(\mathcal{Y}) \times [0,1])$ is also a countable union of compact sets and hence, a Borel set. Noticing that

$$I \subset \bigcup_{\boldsymbol{\gamma} \in \boldsymbol{\pi}^{-1}(\mathcal{X} \setminus (\mathcal{B}^F \cup \mathcal{E}^F))} \boldsymbol{\gamma}([0,1])$$

(the union of the images of the $\boldsymbol{\gamma}$ with $\boldsymbol{\pi}_{\boldsymbol{\gamma}} \in \mathcal{X} \setminus (\mathcal{B}^F \cup \mathcal{E}^F)$), it is enough to finish the proof that we establish $|\boldsymbol{\mu}|(I) = \|\boldsymbol{\mu}\|_{TV}$.

First let us prove that $\Xi(\mathcal{B}^F) = 0$. For this, note that $\mathcal{B}^0 \cap \mathcal{C}^{\mathcal{J}} \cap \mathcal{X}$ is empty whence, by (5),

$$\beta(\mathbb{R}^N \setminus \mathfrak{e}(\mathcal{X})) = \Xi(\mathfrak{b}^{-1}(\mathbb{R}^N \setminus \mathfrak{e}(\mathcal{X})) \setminus \mathcal{C}^{\mathcal{J}}) = \Xi(\mathcal{B}^0 \setminus \mathcal{C}^{\mathcal{J}}) = \Xi(\mathcal{B}^0).$$

Analogously, one has

$$\epsilon(\mathbb{R}^N \setminus \mathfrak{e}(\mathcal{X})) = \Xi(\mathfrak{e}^{-1}(\mathbb{R}^N \setminus \mathfrak{e}(\mathcal{X})) \setminus \mathcal{C}^{\mathcal{J}}) \leq \Xi(\mathcal{C}_1 \setminus \mathcal{X}) = 0$$

and, since $\epsilon = \beta$, we conclude that $\Xi(\mathcal{B}^0) = 0$. Now, assume for $n > 0$ that $\Xi(\mathcal{B}^{n-1}) = 0$. Then, by (5) again and the fact that $\beta = \epsilon$,

$$\begin{aligned}\Xi(\mathcal{B}^n) &= \Xi(\mathcal{B}^n \setminus \mathcal{C}^{\mathcal{J}}) = \beta(\mathfrak{e}(\mathcal{B}^{n-1}) \setminus \mathfrak{e}(\mathcal{X} \setminus \mathcal{B}^{n-1})) = \epsilon(\mathfrak{e}(\mathcal{B}^{n-1}) \setminus \mathfrak{e}(\mathcal{X} \setminus \mathcal{B}^{n-1})) \\ &= \Xi(\mathfrak{e}^{-1}(\mathfrak{e}(\mathcal{B}^{n-1}) \setminus \mathfrak{e}(\mathcal{X} \setminus \mathcal{B}^{n-1})) \setminus \mathcal{C}^{\mathcal{J}}) \leq \Xi(\mathcal{B}^{n-1}) = 0.\end{aligned}$$

Thus, by induction, $\Xi(\mathcal{B}^F) = 0$. An analogous argument shows that $\Xi(\mathcal{E}^F) = 0$. Note that for all $\boldsymbol{\pi}_{\boldsymbol{\gamma}}$ we have $\|\boldsymbol{\pi}_{\boldsymbol{\gamma}}\|_{TV} = 1$, and also that $\|\boldsymbol{\mu}\|_{TV} = \Xi(\mathcal{X})$. Hence, by applying the right equation of (4) to I, we obtain

$$\begin{aligned}|\boldsymbol{\mu}|(I) &= \int_{\mathcal{C}_1} |\boldsymbol{\pi}_{\boldsymbol{\gamma}}|(I)\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}) \geq \int_{\mathcal{Y}} |\boldsymbol{\pi}_{\boldsymbol{\gamma}}|(I)\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}) = \int_{\mathcal{Y}} \|\boldsymbol{\pi}_{\boldsymbol{\gamma}}\|\, d\Xi(\boldsymbol{\pi}_{\boldsymbol{\gamma}}) \\ &= \Xi(\mathcal{Y}) = \Xi(\mathcal{X} \setminus (\mathcal{B}^F \cup \mathcal{E}^F)) = \Xi(\mathcal{X}) = \|\boldsymbol{\mu}\|_{TV}.\end{aligned}$$

□

4. Uniqueness of inverse problem

In this section, we will consider a linear operator $\mathfrak{A} : \mathcal{M}(S)^N \to \mathcal{H}$, where $\mathcal{H}$ is a real Hilbert space, that is continuous in the weak-star topology on $\mathcal{M}(S)^N$. By the Banach-Alaoglu Theorem, any TV-norm bounded sequence contains a weak-star convergent subsequence and, hence, the weak-star continuity of $\mathfrak{A}$ and the separability of $C(S)$ (which entails the metrizability of $\mathcal{M}(S)^N$ with the weak-star topology) imply that $\mathfrak{A}$ is compact; i.e., $\mathfrak{A}$ maps TV-norm bounded sets into precompact sets in $\mathcal{H}$.

The focus of this paper is on the inverse problem associated with $\mathfrak{A}$; that is, estimate $\boldsymbol{\mu}$ given information about $\mathfrak{A}\boldsymbol{\mu}$. It is well-known that the compactness of $\mathfrak{A}$ makes this an ill-posed problem and moreover, for our main motivation of magnetic source recovery problems, $\mathfrak{A}$ is not even injective. It is classical to use regularization methods to produce 'good' approximate solutions to $\mathfrak{A}\boldsymbol{\mu} = f$ by adjoining a penalty functional. In [6] and [5], we considered regularization using the TV-norm and showed this is consistent for recovering a certain class of measures including sparse measures, e.g., those with finite support (see discussion below). Specifically, we consider for $f \in \mathcal{H}$ and $\lambda > 0$, the functional

$$\mathfrak{F}_{\lambda,f}(\boldsymbol{\mu}) := \|f - \mathfrak{A}\boldsymbol{\mu}\|_{\mathcal{H}}^2 + \lambda\|\boldsymbol{\mu}\|_{TV}. \tag{6}$$

Note that, since $\mathfrak{F}_{\lambda,f}(0) = \|f\|_{\mathcal{H}}^2 < \infty$, then, by the Banach–Alaoglu theorem, $\mathfrak{F}_{\lambda,f}$ has a minimizer on the closed ball $\{\|\boldsymbol{\mu}\|_{TV} \leq \|f\|_{\mathcal{H}}^2/\lambda\} \subset \mathcal{M}(S)^N$.

Remark 2. Recall that a measure $\boldsymbol{\mu} \in \mathcal{M}(S)^N$ minimizes $\mathfrak{F}_{\lambda,f}$ if an only if it satisfies (*cf* for instance [10])

$$\begin{array}{lll} \mathfrak{A}^*(f - \mathfrak{A}\boldsymbol{\mu}) & = \frac{\lambda}{2}\boldsymbol{u}_{\boldsymbol{\mu}} & |\boldsymbol{\mu}|\text{-a.e. and} \\ |\mathfrak{A}^*(f - \mathfrak{A}\boldsymbol{\mu})| & \leq \frac{\lambda}{2} & \text{everywhere on } S. \end{array} \tag{7}$$

Moreover, since $\mathfrak{F}_{\lambda,f}$ is strictly convex, if $\boldsymbol{\mu}' \in \mathcal{M}(S)^N$ is another minimizer of $\mathfrak{F}_{\lambda,f}$ then $\mathfrak{A}(\boldsymbol{\mu}' - \boldsymbol{\mu}) = 0$.

Let $\mathcal{D}^*(\mathbb{R}^N)$ denote the space of distributions on $\mathbb{R}^N$. The following lemma is straightforward, so we omit the proof. Its assumptions are in particular satisfied in the setting of inverse magnetization problems on slender samples, see [6].

Lemma 4.1. *Let $S, Q \subset \mathbb{R}^N$ be compact, ρ a positive measure with $\operatorname{supp}\rho = Q$ and r the minimal distance between S and Q. Also take an injective and bounded integral operator $\mathfrak{G} : \mathcal{D}^*(\mathbb{R}^N) \longrightarrow L^2(Q,\rho)^M$ with an integration kernel $\boldsymbol{G}(x, y)$ that has bounded derivatives and is C^2 on $\{|x - y| \geq r\}$. Let for $x \in Q$ and $\boldsymbol{\mu} \in \mathcal{M}(S)^N$,*

$$\mathfrak{A}(\boldsymbol{\mu})(x) := \mathfrak{G}(\nabla\cdot\boldsymbol{\mu})(x) = -\int D_y\boldsymbol{G}(x, y)\, d\boldsymbol{\mu}(y),$$

where $D_y\boldsymbol{G}(x, y)$ is the differential of $\boldsymbol{G}$ with respect to y and the integration is done using matrix multiplication. Then,

(i) $\mathfrak{A}$ is weak-star continuous and its kernel consists of divergence free measures on $\mathcal{M}(S)^N$,

(ii) letting $O := \{x \in \mathbb{R}^N : |x-y| < r \text{ for some } y \in S\}$ *and, for every* $\boldsymbol{f} \in L^2(Q,\rho)^M$,

$$g(y) := -\int \boldsymbol{G}(x,y)\cdot \boldsymbol{f}(x)\, d\rho(x), \text{ for } y \in O,$$

we get that $g \in C^2(O)$ *and* ∇g *extends* $\mathfrak{A}^*\boldsymbol{f}$.

The theorem below is our main result, and its assumptions are satisfied in slender sample cases of the inverse magnetization problem, but we state the result in greater generality as allowed by the argument of the proof.

Theorem 4.2. *Let* $S \subset \mathbb{R}^N$ *be compact and* $\mathfrak{A} : \mathcal{M}(S)^N \to \mathcal{H}$ *be weak-star continuous. Assume the following*

(i) the kernel of $\mathfrak{A}$ *consists of divergence free measures on* $\mathcal{M}(S)^N$,

(ii) for every $h \in \mathcal{H}$, *there exist an open neighbourhood of* S, *say* $O \subset \mathbb{R}^N$, *together with* $g \in C^2(O)$ *such that* ∇g *extends* $\mathfrak{A}^*h$ *to* O

Then for every $f \in \mathcal{H}$ *and* $\lambda > 0$, *the functional* $\mathfrak{F}_{\lambda,f}$ *has a unique minimizer over* $\mathcal{M}(S)^N$.

Proof. Assume for a contradiction that $\boldsymbol{\mu}_\lambda$ and $\widetilde{\boldsymbol{\mu}_\lambda}$ are two distinct minimizers of $\mathfrak{F}_{\lambda,f}$ and let $\boldsymbol{\mu} := \widetilde{\boldsymbol{\mu}_\lambda} - \boldsymbol{\mu}_\lambda$. As $\boldsymbol{\mu}$ is absolutely continuous with respect to $|\boldsymbol{\mu}|$, the Radon-Nykodim decompositions of $\boldsymbol{\mu}_\lambda$ and $\widetilde{\boldsymbol{\mu}_\lambda}$ with respect to $|\boldsymbol{\mu}|$ must have the same singular term. That is, these decomposition are necessarily of the form

$$d\boldsymbol{\mu}_\lambda = \boldsymbol{\phi} d|\boldsymbol{\mu}| + d\boldsymbol{\nu}, \qquad d\widetilde{\boldsymbol{\mu}_\lambda} = \widetilde{\boldsymbol{\phi}} d|\boldsymbol{\mu}| + d\boldsymbol{\nu},$$

where $|\boldsymbol{\nu}|$ is singular with respect to $|\boldsymbol{\mu}|$ and $\boldsymbol{\phi}$, $\widetilde{\boldsymbol{\phi}}$ are $|\boldsymbol{\mu}|$-integrable $\mathbb{R}^N$-valued functions.

Put for simplicity $\psi = (2/\lambda)(f - \mathfrak{A}(\boldsymbol{\mu}_\lambda)) = (2/\lambda)(f - \mathfrak{A}(\widetilde{\boldsymbol{\mu}_\lambda}))$. Thanks to (7) we know that $\boldsymbol{u}_{\boldsymbol{\mu}_\lambda} = \mathfrak{A}^*\psi$ and $\boldsymbol{u}_{\widetilde{\boldsymbol{\mu}_\lambda}} = \mathfrak{A}^*\psi$, $\boldsymbol{\mu}_\lambda$ and $\boldsymbol{\mu}'_\lambda$-a.e. respectively. Now, since $d|\boldsymbol{\mu}_\lambda| = |\boldsymbol{\phi}| d|\boldsymbol{\mu}| + d|\boldsymbol{\nu}|$ and $d|\widetilde{\boldsymbol{\mu}_\lambda}| = |\widetilde{\boldsymbol{\phi}}| d|\boldsymbol{\mu}| + d|\boldsymbol{\nu}|$, we have that

$$\boldsymbol{u}_{\boldsymbol{\mu}} d|\boldsymbol{\mu}| = d\boldsymbol{\mu} = \boldsymbol{u}_{\widetilde{\boldsymbol{\mu}_\lambda}} d|\widetilde{\boldsymbol{\mu}_\lambda}| - \boldsymbol{u}_{\boldsymbol{\mu}_\lambda} d|\boldsymbol{\mu}_\lambda| = \mathfrak{A}^*\psi d|\widetilde{\boldsymbol{\mu}_\lambda}| - \mathfrak{A}^*\psi d|\boldsymbol{\mu}_\lambda| = \mathfrak{A}^*\psi(|\widetilde{\boldsymbol{\phi}}| - |\boldsymbol{\phi}|) d|\boldsymbol{\mu}|.$$

Therefore $\boldsymbol{u}_{\boldsymbol{\mu}} = \mathfrak{A}^*\psi(|\widetilde{\boldsymbol{\phi}}| - |\boldsymbol{\phi}|)$ at $|\boldsymbol{\mu}|$-a.e point, and since $|\mathfrak{A}^*\psi| \le 1$ everywhere, by (7), it holds that $\boldsymbol{u}_{\boldsymbol{\mu}}(x) = \pm_x \mathfrak{A}^*\psi(x)$ for $|\boldsymbol{\mu}|$-a.e. x, where the choice of sign $\pm_x$ has a subscript x to indicate that it may vary with x.

From remark 2 we know that $\mathfrak{A}\boldsymbol{\mu} = 0$. Thus, by assumption *(i)*, $\boldsymbol{\mu}$ is divergence free and, if we let $X = \{x \in \operatorname{supp}|\boldsymbol{\mu}| : \boldsymbol{u}_{\boldsymbol{\mu}}(x) = \pm_x \mathfrak{A}^*\psi(x)\}$, then using Lemma 3.2 we can find a Lipschitz continuous function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^N$ such that, for a.e. $t \in \mathbb{R}$, $\boldsymbol{f}(t) \in X$ and $\boldsymbol{f}'(t) = \boldsymbol{u}_{\boldsymbol{\mu}}(\boldsymbol{f}(t))$. Since $\operatorname{supp}\boldsymbol{\mu}$ is closed at it contains X, in fact we get that $\boldsymbol{f}(\mathbb{R}) \subset \operatorname{supp}\boldsymbol{\mu}$, and thus $\boldsymbol{f}(\mathbb{R})$ is compact.

Also, using now assumption *(ii)*, there exist an open neighborhood of S, $O \subset \mathbb{R}^N$, and a $g \in C^1(O)$ such that $\nabla g = \mathfrak{A}^*\phi$ on S. Next will show that

$$\text{either} \qquad \boldsymbol{f}' = (\mathfrak{A}^*\psi)\circ \boldsymbol{f} \qquad \text{or} \qquad \boldsymbol{f}' = -(\mathfrak{A}^*\psi)\circ \boldsymbol{f}. \tag{8}$$

In order to establish (8), by the connectedness of $\mathbb{R}$, it is enough to prove that for any $t_0 \in \mathbb{R}$ there exists an open interval $I_0 \subset \mathbb{R}$ containing t_0 and on which (8) is satisfied.

Since $|\nabla g| = |\mathfrak{A}^*\psi| = 1$ on the support of $\boldsymbol{\mu}_\lambda$, which contains the image of $\boldsymbol{f}$, there exist a $i \in \{1, 2, ..., N\}$ such that $\partial_i g(\boldsymbol{f}(t_o)) \neq 0$. Now, by the method of characteristics (see for example [13, section 3.2.5]), there exist a neighborhood V of $\boldsymbol{f}(t_o)$ and, for $j \in \{1, 2, ..., N\} \setminus i$, C^1 functions F_j on V that satisfy,

$$\begin{cases} \nabla F_j \cdot \nabla g = 0 & \text{on } V \\ F_j(x) = x_j & \text{on } V \cap \{\boldsymbol{f}(t_0) + x \mid x_i = 0\}. \end{cases}$$

We further let $F_i = g$. Then, using the inverse function theorem, we can find a connected neighborhood of $\boldsymbol{f}(t_0)$, $U \subset V$ such that the function $\boldsymbol{F} := (F_1, F_2, ..., F_N)$ is bi-Lipschitz on U and $[D\boldsymbol{F}(x)]^t \nabla g(x) = |\nabla g(x)|^2 \mathbf{e}_i$, with superscript t to mean "transpose". Now, since for a.e. $t \in \mathbb{R}$, $\boldsymbol{f}(t) \in X$, we get that

$$\pm_{\boldsymbol{f}(t)} \nabla g(\boldsymbol{f}(t)) = \pm_{\boldsymbol{f}(t)} [\mathfrak{A}^*\psi](\boldsymbol{f}(t)) = \boldsymbol{u}_{\boldsymbol{\mu}}(\boldsymbol{f}(t)) = \boldsymbol{u}_{\boldsymbol{\pi}_{\boldsymbol{f}}}(\boldsymbol{f}(t)) = \boldsymbol{f}'(t), \quad \text{for a.e. } t \in \mathbb{R}.$$

Thus, $(\boldsymbol{F} \circ \boldsymbol{f})'(t) = [D\boldsymbol{F}(\boldsymbol{f}(t))]^t \boldsymbol{f}'(t) = \pm_{\boldsymbol{f}(t)} \mathbf{e}_i$ for a.e. $t \in \mathbb{R}$. Therefore, the fact that

$$(\boldsymbol{F} \circ \boldsymbol{f})(t_2) - (\boldsymbol{F} \circ \boldsymbol{f})(t_1) = \int_{t_1}^{t_2} (\boldsymbol{F} \circ \boldsymbol{f})'(t) dt \quad \text{whenever } \boldsymbol{f}([t_1, t_2]) \subset U \text{ and } t_1 \leq t_2$$

implies that

$$(\boldsymbol{F} \circ \boldsymbol{f})(\boldsymbol{f}^{-1}(U)) \subset \{(\boldsymbol{F} \circ \boldsymbol{f})(t_0) + t\mathbf{e}_i\}. \tag{9}$$

Let I_0 be an open interval such that $t_0 \in I_0$ and that its closure, $\overline{I_0}$, is contained in $\boldsymbol{f}^{-1}(U)$. We will show that $\boldsymbol{\gamma} := \boldsymbol{F} \circ \boldsymbol{f}|_{I_0}$ is injective by showing that the set

$$E := \{x \in \boldsymbol{\gamma}(I_0) : \#(\boldsymbol{\gamma}, x) > 1\}$$

is empty. First, we will show that for η^1-a.e. $x \in E$, there exists a $t_x \in \boldsymbol{\gamma}^{-1}(x)$ such that $\boldsymbol{\gamma}'(t_x) \neq \pm_{\boldsymbol{f}(t_x)} \mathbf{e}_i$. For this, fix $x \in E$ and take any $t_1 \in \boldsymbol{\gamma}^{-1}(x)$. We may assume that x is a regular value of $\boldsymbol{\gamma}$, by Sard's theorem for Lipschitz functions [21, Theorem 7.6].

If $t_1 = \max(\boldsymbol{\gamma}^{-1}(x) \setminus \{t_1\})$ rename t_1 by t_2 and let $t_1 = \max\{t \in \boldsymbol{\gamma}^{-1}(x) : t < t_2\}$. Otherwise, let $t_2 = \min\{t \in \boldsymbol{\gamma}^{-1}(x) : t_1 < t\}$. Note that if $t_1 = t_2$ and $\boldsymbol{\gamma}'(t_1)$ is well defined, then $\boldsymbol{\gamma}'(t_1) = 0$. Hence, for $t_1 = t_2$ we can let $t_x = t_1$. Now assume that $t_1 < t_2$. In view of (9), for all $t \in [t_1, t_2]$ there exists a $s(t) \neq 0$ such that, $\boldsymbol{\gamma}(t) - \boldsymbol{\gamma} p(t_2) = \boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(t_1) = s(t)\mathbf{e}_i$. Hence, the continuity of $\boldsymbol{\gamma}$ implies that the function s does not change sign in (t_1, t_2) and thus, if both $\boldsymbol{\gamma}'(t_1)$ and $\boldsymbol{\gamma}'(t_2)$ are defined, it holds that $\boldsymbol{\gamma}'(t_1) = -\boldsymbol{\gamma}'(t_2)$. Therefore, as x is such that $\boldsymbol{\gamma}$ is differentiable at every point of $\boldsymbol{\gamma}^{-1}(x)$ because it is a regular value, we get for $t_1 < t_2$ that either $t_x = t_1$ or $t_x = t_2$ satisfies $\boldsymbol{\gamma}'(t_x) \neq \pm_{\boldsymbol{f}(t_x)} \mathbf{e}_i$.

Now, since for a.e. $t \in \mathbb{R}$, $(\boldsymbol{F} \circ \boldsymbol{f})'(t) = \pm_{\boldsymbol{f}(t)} \mathbf{e}_i$, the set $\{t_x : x \in E\}$ has 1-dimensional Lebesgue measure zero. Hence, since $\boldsymbol{\gamma}$ is Lipschitz continuous with Lipschitz constant 1 and $E = \boldsymbol{\gamma}(\{t_x : x \in E\})$, we get that $\eta^1(E) = 0$ as well. Finally, using this result with (9) and the intermediate value theorem on $F_i \circ \boldsymbol{f}$, we get that E is indeed empty, and thus, $\boldsymbol{\gamma}$ is injective.

Then, the injectiveness of $\boldsymbol{\gamma}$ implies that either $\boldsymbol{\gamma}(t) = (t_0 - t)\mathbf{e}_i$ or $\boldsymbol{\gamma}(t) = (t - t_0)\mathbf{e}_i$ on I_0. Therefore, $\pm_{\boldsymbol{f}(t)}$ is constant on I_0 and thus, as we mentioned above, the connectedness of $\mathbb{R}$ implies that $\pm_{\boldsymbol{f}(t)}$ is constant on $\mathbb{R}$. However, for any two $t_1, t_2 \in \mathbb{R}$,

$$(g \circ \boldsymbol{f})(t_2) - (g \circ \boldsymbol{f})(t_1) = \int_{t_1}^{t_2} \nabla g(\boldsymbol{f}(t)) \cdot \boldsymbol{f}'(t) dt = \pm \int_{t_1}^{t_2} |\nabla g(\boldsymbol{f}(t))|^2 dt = \pm(t_2 - t_1)$$

which is not possible since g is continuous and hence bounded on the compact set $\overline{\boldsymbol{f}(\mathbb{R})}$. □

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