Inverse problem for the Helmholtz equation and singular sources in the divergence form

L. Baratchart[1], H. Haddar[2], C. Villalobos Guillén[1,*]
[1]INRIA d'Université Côte d'Azur, Sophia Antipolis, France
[2]INRIA Saclay & ENSTA Paris Tech, Palaiseau, France
[*]Email: cristobal.villalobos-guillen@inria.fr

Abstract

We shall discuss an inverse problem where the underlying model is related to sources generated by currents on an anisotropic layer. This problem is a generalization of another motivated by the recovering of magnetization distribution in a rock sample from outer measurements of the generated static magnetic field. The original problem can be formulated as inverse source problem for the Laplace equation [1,2] with sources being the divergence of the magnetization whereas the generalization comes from taking the Helmholtz equation. Either inverse problem is non uniquely solvable with a kernel of infinite dimension. We shall present a decomposition of the space of sources that will allow us to discuss constraints that may restore uniqueness and propose regularization schemes adapted to these assumptions. We then present some validating experiments and some related open questions.

Keywords: Helmholtz equation, Divergence free, Inverse problems, Magnetization

1 Introduction

Let $k \geq 0$, $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $S \subset \partial\Omega$ be compact, and $G(x) = -\frac{e^{ik|x|}}{4\pi|x|}$, a Green function for the Helmholtz equation. Let $H^s(\partial\Omega)$ denote the Bessel potential space of order s and $H^s_1(\partial\Omega)$, the space of 1 forms on $\partial\Omega$ with coefficients in $H^s(\partial\Omega)$.

Given a $\mathbf{M} \in L_2(\partial\Omega)^3$, with support contained in S, define $\mathcal{J}(\mathbf{M}) := G * \nabla\cdot(\mathbf{M}\sigma)$, where σ denotes the surface measure on $\partial\Omega$. It follows that $\mathcal{J}(\mathbf{M})$ is analytic on $\mathbb{R}^3 \setminus S$ and locally integrable. Then, for $w = \mathcal{J}(\mathbf{M})$,

$$\Delta w + k^2 w = \nabla \cdot (\mathbf{M}\sigma). \tag{1}$$

Note that this implies that the kernel of $\mathcal{J}$ consist precisely of all $\mathbf{M}$ such that $\nabla \cdot (\mathbf{M}\sigma) = 0$. Also, $w = \mathcal{J}(\mathbf{M})$ satisfies the Sommerfeld radiation condition:

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial|x|} - ik \right) w(x) = 0. \tag{2}$$

If we let $\boldsymbol{\nu}$ denote the normal vector to $\partial\Omega$, we can write $\mathbf{M} = \boldsymbol{\nu} M_{\boldsymbol{\nu}} + \mathbf{M}_T$, with $M_{\boldsymbol{\nu}} := \mathbf{M}\cdot\boldsymbol{\nu}$ and $\mathbf{M}_T := \mathbf{M} - \boldsymbol{\nu} M_{\boldsymbol{\nu}}$. Note that in this case $\mathbf{M}_T(x)$ is tangent to $\partial\Omega$ for σ-a.e. $x \in \partial\Omega$ and we can define is tangential divergence $\nabla_T \cdot \mathbf{M}_T$ weakly. We will denote by $\mathbf{M}_T^\flat$ the 1-form that is the flat of $\mathbf{M}_T$, i.e., for every $p \in \partial\Omega$ and v tangent to $\partial\Omega$ at p, $\mathbf{M}_T^\flat(p)(v) = \mathbf{M}_T(p) \cdot v$.

We will restrict ourselves to the case when $M_{\boldsymbol{\nu}} \in H^{1/2}(\partial\Omega)$ and $\nabla_T \cdot \mathbf{M}_T \in H^{-1/2}(\partial\Omega)$. Noticing that, if $\mathbf{M}_T^\flat \in H_1^{1/2}(\partial\Omega)$ then $\nabla_T \cdot \mathbf{M}_T \in H^{-1/2}(\partial\Omega)$, we will represent a source as a pair $(\mathbf{M}_T^\flat, M_{\boldsymbol{\nu}}) \in \mathcal{M} := H_1^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$. With a slight abuse of notation we will identify $\mathbf{M}$ with the pair $(\mathbf{M}_T^\flat, M_{\boldsymbol{\nu}})$, and think of $\mathcal{M}$ as a subspace of $L_2(\partial\Omega)^3$.

Let SL and DL denote the single and double layer potentials associated to (1). That is, for $x \in \mathbb{R}^3 \setminus \partial\Omega$, $\psi \in H^{-1/2}(\partial\Omega)$ and $\phi \in H^{1/2}(\partial\Omega)$,

$$SL\psi(x) := \int_{\partial\Omega} G(x-y)\psi(y)d\sigma(y),$$
$$DL\phi(x) := \int_{\partial\Omega} \partial_{\boldsymbol{\nu},y} G(x-y)\phi(y)d\sigma(y),$$

where $\partial_{\boldsymbol{\nu},y}$ is the normal derivative with respect to the variable y. Then we have that $\mathcal{J}(\mathbf{M}) = -DL(M_{\boldsymbol{\nu}}) + SL(\nabla_T \cdot \mathbf{M}_T)$.

The following result shows the limitations for the theoretical inverse problem. Given $Q \subset \mathbb{R}^3 \setminus S$ compact and a measure ρ supported on Q, define $A_Q : \mathcal{M} \to L_2(Q, \rho)$ as $A_Q(\mathbf{M}) = \mathcal{J}(\mathbf{M})|_Q$.

Theorem 1 *Take D a Lipschitz domain containing S with unbounded complement, and $Q \subset \mathbb{R}^3 \setminus S$ compact. If $\mathbb{R}^3 \setminus S$ is connected and either*

1. *∂D is an analytic surface and $\overline{Q} \cap \partial D$ has Hausdorff dimension>1,*
2. *or $\partial D \subset \overline{Q}$,*

then, $\mathbf{M}$ belong to the kernel of A_Q if and only if $M_{\boldsymbol{\nu}} = 0$ and $\nabla_T \cdot \mathbf{M}_T = 0$.

2 Full Bounded Lipschitz Domain

In what follows we will assume that $S = \partial\Omega$. Let $j(\phi,\psi) = -DL(\phi) + SL(\psi)$, for $(\phi,\psi) \in H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)$, and define for $\phi \in H^{1/2}(\partial\Omega)$ and $\psi \in H^{-1/2}(\partial\Omega)$;

$$\begin{aligned} S\psi(x) &:= \text{p.v.}\int_{\partial\Omega} G(x-y)\psi(y)d\sigma(y), \\ K\phi(x) &:= \text{p.v.}\int_{\partial\Omega} \partial_{\boldsymbol{\nu},y} G(x-y)\phi(y)d\sigma(y), \\ T\phi &:= (\partial_{\boldsymbol{\nu}} DL\phi)|_{\partial\Omega}. \end{aligned}$$

Next, let $P_+, P_- : H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)$ be defined by matrix multiplication as

$$P_+(\phi,\psi) := \begin{pmatrix} \frac{1}{2}Id - K & S \\ -T & \frac{1}{2}Id + K' \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

$$P_-(\phi,\psi) := \begin{pmatrix} \frac{1}{2}Id + K & -S \\ T & \frac{1}{2}Id - K' \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Then P_+, P_- are Calderón projections [4], they satisfy $P_+ = Id - P_-$, and we obtain the following result:

Theorem 2 *Let $w = \mathcal{J}(\mathbf{M})$. Then, $w|_{\mathbb{R}^3\setminus\overline{\Omega}} = 0$ if and only if $P_+(M_{\boldsymbol{\nu}}, \nabla_T\cdot\mathbf{M}_T) = 0$, and, $w|_\Omega = 0$ if and only if $P_-(M_{\boldsymbol{\nu}}, \nabla_T\cdot\mathbf{M}_T) = 0$.*

3 Decomposition of $\mathcal{M}$

Let $\omega \in H^1_{loc}(\mathbb{R}^3\setminus\Omega)$ satisfy (2) and

$$\begin{aligned} \Delta\omega + k^2\omega = 0 \text{ on } \mathbb{R}^3\setminus\Omega \\ \omega = \left(K + \frac{1}{2}Id\right)1_{\partial\Omega} \text{ on } \partial\Omega, \end{aligned}$$

where $1_{\partial\Omega}$ is the constant function with value 1 in $\partial\Omega$, and let $f = \overline{\partial_{\boldsymbol{\nu}}\omega}$. Then can define the following subspaces of $\mathcal{M}$

$$\mathcal{M}_0 = \{\mathbf{M}\in\mathcal{M} \ : \ (M_{\boldsymbol{\nu}}, \nabla_T\cdot\mathbf{M}_T) = (0,0)\},$$

$$\mathcal{M}_+ = \{\mathbf{M}\in\mathcal{M}_0^\perp\cap\mathcal{M} \ : \ P_+(M_{\boldsymbol{\nu}}, \nabla_T\cdot\mathbf{M}_T) = (0,0)\},$$

$$\mathcal{M}_- = \{\mathbf{M}\in\mathcal{M}_0^\perp\cap\mathcal{M} \ : \ P_-(M_{\boldsymbol{\nu}}, \nabla_T\cdot\mathbf{M}_T) = (0,0)\},$$

where $\mathcal{M}_0^\perp$ is taken in $L_2(\partial\Omega)^3$. Also, there exists a $\mathbf{M}_0$ that satisfies for every $\mathbf{M}\in H^{1/2}(\partial\Omega)^3$,

$$\begin{aligned} (\mathbf{M}_0, \mathbf{M})_{H^{1/2}(\partial\Omega)} &= (f, \mathbf{M}\cdot\boldsymbol{\nu})_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \\ &- (\overline{K1_{\partial\Omega}}, \nabla_T\cdot\mathbf{M}_T)_{H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)}, \end{aligned}$$

which allows us to define $\mathcal{M}_H$ =vect$\{\mathbf{M}_0\}$.

Let us observe for instance that when Ω is a ball then $\mathbf{M}_0 = c\nu$ where c is a constant.

Theorem 3 *$\mathcal{M}_0$, $\mathcal{M}_+\oplus\mathcal{M}_-$ and $\mathcal{M}_H$ are pairwise orthogonal as subspaces of $\mathcal{M}$ and*

$$\mathcal{M} = \mathcal{M}_0\oplus\mathcal{M}_+\oplus\mathcal{M}_-\oplus\mathcal{M}_H.$$

Furthermore, when k^2 is a Neumann eigenvalue for $-\Delta$ in Ω and the trace of one eigenfunction on $\partial\Omega$ coincides with $1_{\partial\Omega}$, then $\mathbf{M}_0 = 0$, and hence $\mathcal{M} = \mathcal{M}_0\oplus(\mathcal{M}_+ + \mathcal{M}_-)$.

4 Partial inversion of the problem

Using the above decomposition it is now clear that the problem of recovering sources when measurements are only done outside of the sample is only solvable up to an element of $\mathcal{M}_0\oplus\mathcal{M}_+$.

Given an original source $\mathbf{M}$, let $\tilde{\mathbf{M}}$ denote the unique element of $\mathcal{M}_-\oplus\mathcal{M}_H$ which generates the same potential as $\mathbf{M}$ outside of Ω. We will now give a rough description of a way to obtain $\tilde{\mathbf{M}}$. Let $Q\subset\mathbb{R}^3\setminus\overline{\Omega}$ be such that for every $w\in\mathcal{J}(\mathcal{M})$, $w|_Q = 0$ implies $w|_{\mathbb{R}^3\setminus\overline{\Omega}} = 0$ (for example, a dense subset of some type of analytic surface). Since, $\mathcal{J}(\mathbf{M})|_{\mathbb{R}^3\setminus\overline{\Omega}} = 0$ implies $P_+(M_{\boldsymbol{\nu}}, \nabla_T\cdot\mathbf{M}_T) = 0$, then the linear operator B, defined from $H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\overline{\Omega})\times H^{-1/2}(\partial\Omega)\times L^1_{loc}(\mathbb{R}^3\setminus\Omega)^3$, that sends (ϕ,ψ) to $(P_-(\phi,\psi), j(\phi,\psi)|_Q)$, is injective. Letting $\omega = \mathcal{J}(\mathbf{M})$, then $B^{-1}(0,\omega)$ is well defined and such that $P_+(B^{-1}(0,\omega) - (\tilde{M}_{\boldsymbol{\nu}}, \nabla_T\cdot\tilde{\mathbf{M}}_T)) = 0$. Hence, using the definition of $\mathbf{M}_0$ and the above results, we can recover $\tilde{\mathbf{M}}$.

References

[1] L. Baratchart, C. Gerhards, and A. Kegeles, Decomposition of L^2-Vector Fields on Lipschitz Surfaces: Characterization via Null-Spaces of the Scalar Potential, *SIAM J. Math. Anal.* **53(4)** (2021), pp. 4096–4117.

[2] L. Baratchart, C. Villalobos-Guillen, D. P. Hardin, M. C. Northington, E. B. Saff, Inverse Potential Problems for Divergence of Measures with Total Variation Regularization, *Found. Comp. Math.* **20** (2020), pp. 1273–1307.

[3] L. Baratchart, H. Haddar, C. Villalobos-Guillen, On the Helmholtz equation and singular sources in divergence form, *[Unpublished manuscript]*

[4] W. McLean, *Strongly Elliptic Systems And Boundary Integral Equations.*, Cambridge University Press, Cambridge, 2000.